

Systematic Study of the Single Instanton Approximation in QCD

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Single-instanton approximation (SIA) is often used to evaluate analytically instanton contributions euclidean correlation function in QCD at small distances. We discuss how this approximation can be consistently derived from the theory of instanton ensemble and give precise definitions to a number of different “quark effective masses”, generalizing the parameter m^* , which was introduced long ago to account for the collective contribution of the whole ensemble. We test numerically the range of applicability of the SIA for different quantities. Furthermore, we determine all the effective masses (for random and interacting instanton liquid models) as well as from phenomenology, and discuss to what extent those are universal.

I. INTRODUCTION

The instanton liquid model of the QCD vacuum [1] is based on a semiclassical approximation, in which all gauge configurations are replaced by an ensemble of topologically non-trivial fields, instantons and anti-instantons. It remains a model because we do not still understand why large-size instantons are not present in the ensemble. Fits to phenomenology and later lattice studies showed that their total density is $n_0 \simeq 1 fm^{-4}$ while the typical size of about $\rho \sim 1/3 fm$, leading to small diluteness parameter $n_0 \rho^3 \sim 10^{-2}$ [1]. With these parameters, the model quantitatively explains such important phenomena as spontaneous $SU(N_f)$ chiral symmetry breaking for N_f quark flavors, the explicit U(1) symmetry breaking, and many more other details of hadronic correlators and spectroscopy (for a recent example see discussion of vector and axial correlators [2], for a review see [3]). The main feature of the instanton[17] ensemble is that each pseudo-particle is an effective vertex with $2N_f$ quark lines [8], which are exchanged between them and fill the vacuum. A theory is developed, called Interacting Instanton Liquid Model (IILM) which include these 't Hooft interactions *to all orders* [3].

If new sources (external currents) are added, they produce extra quarks which interact with those in vacuum and produce non-trivial correlation functions. In particular, many (Lorentz scalar) chirally odd local operators obtain non-zero vacuum expectation values. In general, all of those “condensates” and correlation functions are determined by the interaction of instantons and thus depend on the global (collective) properties of the ensemble.

On the other hand, as the instanton vacuum is fairly dilute, one may think that the correlation functions at distances short compared to instanton spacing $x \ll R = n^{-1/4} \sim 1 fm$ may be dominated by a *single* instanton, the closest (or leading) one (LI). This framework (which we shall refer to as the “single instanton approximation”, SIA) has the advantage to allow to carry out calculations analytically. It is therefore possible to obtain closed expressions for instanton contribution to Green’s functions in momentum or in Borel space.

In SIA collective contribution of all instantons *other than the leading one* are taken care of by a single effective parameter, usually called effective mass, m^* . In the simplest approximation, it can be associated with an *average* value of the quark condensate [4]:

$$m^* = m - \frac{2}{3} \pi^2 \rho^2 < \bar{u} u >, \quad (1)$$

which leads to the value $m^* \simeq 170 MeV$ [1]. Note that it is already very different from what one infers from the same model for long distance (or zero Euclidean momentum) limit of the quark propagator, which gives *constituent quark mass* of the order of 400 MeV.

Furthermore, although the SIA has been used in several phenomenological studies (e.g. [1], [5, 6, 7], and references therein), its derivation was never discussed in detail, its range of applicability was never quantitatively checked, and the values of relevant effective masses well specified. And indeed, if one uses the value $m^* \simeq 170 MeV$ the correlation functions, evaluated in the SIA, do not agree with the results of the random and interacting instanton liquid [3].

In this paper we identify the origin of such discrepancy and calculate the values of effective mass appropriate for different observables. This analysis reveals that the discrepancy between SIA and full liquid calculations is due to an incorrect estimate of the effective mass, m^* . We also present a systematic study of the SIA in QCD by itself.

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We show that the approach is really accurate only for calculations that involve operators of dimension six or more, or correlators with more than one zero-mode propagator. We shall also prove that the mass terms, appearing in matrix elements involving different numbers of zero-mode propagators, are indeed independent parameters that have to be fixed separately. We provide with the definitions of all such mass factors in terms of averages of the instanton ensembles and prove that they are nearly universal, i.e. the same for all similar correlation functions.

The paper is organized as follows. In section II we derive the SIA from the theory of the instanton ensemble, in section III we present the results of our numerical simulations that estimate the contribution from the leading-instanton to several correlation functions. In section IV, we evaluate the effective mass terms both from the random and interacting instanton liquid and compare it with the values obtained phenomenologically from the pion sum-rule. In section V we compare our effective masses with the so-called “determinantal masses”, which are other effective parameters that can be defined in terms of averages of the fermionic determinant. The main results of our analysis are summarized in section VI.

II. QUARK PROPAGATOR

In this section we review how the quark propagator in the instanton vacuum is obtained and present consistent derivation of the SIA.

The quark propagator in general background field is

$$S_I(x, y) = \langle x | (i \not{D}_I + i m)^{-1} | y \rangle, \quad (2)$$

where \not{D}_I denotes the Dirac operator. The inverse (2) can be formally represented as an expansion in eigenmodes of the Dirac operator:

$$S_I(x, y) = \sum_{\lambda} \frac{\psi_{\lambda}(x) \psi_{\lambda}^{\dagger}(y)}{\lambda + i m}, \quad i \not{D}_I \psi_{\lambda}(x) = \lambda \psi_{\lambda}(x). \quad (3)$$

From eq. (3) it follows that the propagator of light quarks is dominated by eigenmodes with small virtuality.

We begin by considering the academic case in which the vacuum contains only one isolated instanton. One eigenmode of \not{D}_I with zero virtuality (zero-modes) is given by 't Hooft [8], [9]:

$$i \not{D} \psi_0(x) = 0, \quad \psi_{0 \alpha \nu}(x; z) = \frac{\rho}{\pi} \frac{1}{((x - z)^2 + \rho^2)^{3/2}} \cdot \left[\frac{1 - \gamma_5}{2} \frac{\not{x} - \not{z}}{\sqrt{(x - z)^2}} \right]_{\alpha \beta} U_{ab} \epsilon_{\beta b}, \quad (4)$$

where z denotes the instanton position, $\alpha, \beta = 1, \dots, 4$ are spinor indices and U_{ab} represents a general group element.

Isolating the contribution from zero-modes we can write:

$$\begin{aligned} S_I(x, y; z) &= \frac{\psi_0(x - z) \psi_0^{\dagger}(y - z)}{i m} + \sum_{\lambda \neq 0} \frac{\psi_{\lambda}(x - z) \psi_{\lambda}^{\dagger}(y - z)}{\lambda + i m} \\ &= S_I^{zm}(x, y; z) + S_I^{nzm}(x, y; z). \end{aligned} \quad (5)$$

The zero-mode part of the propagator in the field of one instanton can be evaluated from (5) and (4) to give, [10]:

$$S_I^{zm}(x, y; z) = \frac{(\not{x} - \not{z}) \gamma_{\mu} \gamma_{\nu} (\not{y} - \not{z})}{8m} \left[\tau_{\mu}^{-} \tau_{\nu}^{+} \frac{1 - \gamma_5}{2} \right] \phi(x - z) \phi(y - z), \quad (6)$$

where

$$\phi(t) := \frac{\rho}{\pi} \frac{1}{|t| (t^2 + \rho^2)^{3/2}}, \quad \tau_{\mu}^{\pm} := (\tau, \mp i) \quad (7)$$

The corresponding expression in the field of one anti-instanton is obtained through the substitution:

$$\frac{1 - \gamma_5}{2} \longleftrightarrow \frac{1 + \gamma_5}{2} \quad \tau^{-} \longleftrightarrow \tau^{+}. \quad (8)$$

In the chiral limit, $m \rightarrow 0$, the expression for $S_I^{nzm}(x, y; z)$ is also known exactly [11]. In the limit of small distances ($|x - y| \rightarrow 0$), or if the instanton is very far away ($|x - z| \rightarrow \infty$) one has:

$$S_I^{nzm}(x, y; z) \simeq S_0(x, y), \quad (9)$$

where S_0 denotes the free propagator. Typically, corrections to eq. (9) lead to small contributions and will be neglected in what follows. Once the propagator has been calculated, one can in principle evaluate any correlation function in the single-instanton background.

Now, let's turn to the realistic vacuum of QCD. Here, any configuration with a non-zero net topological charge would be highly disfavored by the small value of the θ -angle. Therefore, one is lead to picture the vacuum as an ensemble with equal density of instanton and anti-instantons. If the vacuum is dilute enough, the classical back-ground field can be approximatively taken to be a superposition of separated instantons and anti-instantons [18]:

$$A_\mu(x, \{\Omega_i\}_i) = \sum_I A_\mu^I(x, \{\Omega_i^I\}_i) + \sum_A A_\mu^A(x, \{\Omega_i^A\}_i), \quad (10)$$

where $\{\Omega_i\}_i$ denotes the set of all collective coordinates.

The propagator in such back-ground field can then be evaluated as follows [10]. Let's consider the expansion:

$$S = S_0 + S_0 \not{A} S_0 + S_0 \not{A} S_0 \not{A} S_0 + \dots, \quad (11)$$

where integrations over the positions of each background field insertion is understood. The series (11) can be re-arranged so that all terms depending on the collective coordinates of one instanton field only are summed up first, followed by all terms depending on two instantons and so on. One gets:

$$S = S_0 + \sum_I (S_I - S_0) + \sum_{I \neq J} (S_I - S_0) S_0^{-1} (S_J - S_0) + \dots, \quad (12)$$

where S_I denotes the full propagator in the field of the instanton I so, in the approximation (9) one has:

$$(S_I - S_0)_{ij}(x, y) \simeq \frac{\psi_{0i}^I(x) \psi_{0j}^{\dagger I}(y)}{i m}, \quad (13)$$

where we have dropped all collective coordinates indices. Inserting (13) in (12) and dropping also all spinor indices we get:

$$\begin{aligned} S(x, y) &\simeq S^0(x, y) + \sum_I \frac{\psi_0(x) \psi_0^\dagger(y)}{i m} + \\ &\sum_{I, J} \frac{\psi_{0I}(x)}{i m} \left(\int d^4 z \psi_{0I}^\dagger(z) (i \not{\partial}_z + i m) \psi_{0J}(z) - i m \delta_{IJ} \right) \frac{\psi_{0J}^\dagger(y)}{i m} + \dots, \end{aligned} \quad (14)$$

where $-i m \delta_{IJ}$ has been added in order to relax the $J \neq I$ constraint in the summation. All the terms, starting from the second on, form a geometrical progression, which can be re-summed to give:

$$S(x, y) \simeq S^0(x, y) + \sum_{I, J} \psi_{0I}(x) \left(\frac{1}{T + o(m)} \right)_{IJ} \psi_{0J}^\dagger(y), \quad (15)$$

where T_{IJ} denotes the overlap matrix in zero-modes subspace

$$T_{IJ} = \int d^4 z \psi^\dagger(z)_I (i \not{\partial}) \psi(z)_J. \quad (16)$$

In (15), the zero-mode part the quark propagator is approximatively written as a bilinear form in the space spanned by the quark zero-mode wave functions. From (4) it follows that the contribution coming from all the terms in the sum associated to instantons very far away from the points x and y will be negligible. In particular, the biggest term in (15) is associated to the closest instanton, I^* . Such instanton is dominating if the average of the correlation function calculated retaining only the (I^*, I^*) term in (15) is much larger than the average of the same quantity calculated from all other terms in the sum (15). Notice that this is a much weaker assumption than demanding

$$\psi_{0I^*}(x) \left(\frac{1}{T + o(m)} \right)_{I^* I^*} \psi_{0I^*}^\dagger(y) \gg \sum_{I \neq I^*, J \neq I^*} \psi_{0I}(x) \left(\frac{1}{T + o(m)} \right)_{IJ} \psi_{0J}^\dagger(y), \quad (17)$$

for *each* configuration.

Let us summarize the framework developed so far. First of all, the inverse matrix $(\frac{1}{T})_{IJ}$ contains all the information about the particular configuration of the instanton ensemble. In order to evaluate correlation functions, one needs to average over all possible configurations. Since contributions from distant instantons are suppressed by their zero-mode wave functions, one expects correlation functions with the highest number of zero-modes to be most influenced by the leading-instanton I^* . If it is possible to retain only the contribution from I^* , the global properties of the ensemble are present in the matrix element $(\frac{1}{T})_{I^*I^*}$.

As it was suggested long time ago by one of us [1], one can represent collective contribution of all other instantons, by introducing an effective mass associated to quark propagating in the zero-modes. In other words, one assumes that for $|x - y| < 1fm$, the quark propagator can be written as:

$$S(x, y) = \frac{\psi_0(x)\psi_0^\dagger(y)}{im^*}. \quad (18)$$

With such propagator all quark correlation functions in the instanton back-ground could be evaluated simply by computing all relevant Feynman diagrams and then averaging over the instanton collective coordinates [19].

More specifically, in Random Instanton Liquid Model (RILM) one introduces a model instanton density $n(\rho)$,

$$n_I(\rho) := \bar{n}_I d(\rho), \quad (19)$$

where $\bar{n}_I = \bar{n}_A \simeq \frac{1}{2}fm^{-4}$ and $d(\rho)$ represents the instanton size distribution. The latter is schematically taken to be:

$$d(\rho) = \delta(\rho - \bar{\rho}). \quad (20)$$

with $\bar{\rho} \simeq 1/3fm$. This approach has the advantage to be considerably simple and was also proven to be quite phenomenologically successful [5, 6]. However, we show below that the effective mass defined in (18) is a quantity quite different from its naive estimate (1).

In order to clarify the statement, let us first consider the quark condensate:

$$\chi_{uu} = \langle 0 | Tr \bar{u}(x)u(x) | 0 \rangle = \langle Tr S(x, x) \rangle, \quad (21)$$

where, in general, the average is done over all possible gauge field configurations. In the SIA is easily evaluated:

$$\langle 0 | \bar{u}(x)u(x) | 0 \rangle = \int d^4z \int d\rho \bar{n} d(\rho) \left[\frac{-2\rho^2}{[(z-x)^2 + \rho^2]^3 \pi^2 m_{uu}} \right] \quad (22)$$

where, for reasons that will become clear shortly, we have denoted with m_{uu} the quark effective mass and $\bar{n} := \bar{n}_I + \bar{n}_A$. After performing the integrations one finds:

$$\chi_{uu} = -\frac{\bar{n}}{m_{uu}}, \quad (23)$$

for any normalized $d(\rho)$.

Now, repeating the same calculation in the full liquid [20] gives:

$$\chi_{uu} = \left\langle Tr \left[\sum_{I,J} \psi_{0I}(x) \left(\frac{1}{T} \right)_{IJ} \psi_{0J}^\dagger(x) \right] \right\rangle, \quad (24)$$

where, again, the average is made over all possible configurations of the ensemble. A comparison between (23) and (24) gives:

$$m_{uu} := -\frac{\bar{n}}{\left\langle Tr \left[\sum_{I,J} \psi_{0I}(x) \left(\frac{1}{T} \right)_{IJ} \psi_{0J}^\dagger(x) \right] \right\rangle} \quad (25)$$

Let us now consider another quark condensate:

$$\chi_{uudd} := \langle 0 | Tr [\bar{u}(x)u(x)] \cdot Tr [\bar{d}(x)d(x)] | 0 \rangle = \langle [Tr S(x, x)]^2 \rangle. \quad (26)$$

Such condensate receives double contribution from zero-modes. In the SIA one obtains:

$$\chi_{uudd} = \int d\rho d(\rho) \frac{\bar{n}}{5\pi^2 \rho^4 m_{uudd}^2}, \quad (27)$$

TABLE I: Quark condensates evaluated in the full instanton ensemble and from the leading-instanton, only.

condensate	complete calculation	LI
χ_{uu}	$(-232 \pm 5 MeV)^3$	$(-198 \pm 1 MeV)^3$
χ_{uudd}	$(310 \pm 7 MeV)^6$	$(309 \pm 3 MeV)^6$

where we have now denoted with m_{uudd} the quark effective mass.

Comparing, as before, with the result of full liquid calculations leads to:

$$m_{uudd}^2 = \left(\int d\rho d(\rho) \frac{\bar{n}}{5\pi^2 \rho^4} \right) \frac{1}{\left\langle \left[Tr \sum_{I,J} \psi_{0I}(x) \left(\frac{1}{T} \right)_{IJ} \psi_{0J}^\dagger(x) \right]^2 \right\rangle}} \quad (28)$$

Now, if the effective mass is universal, $(m_{uu})^2 = m_{uudd}^2$, it would imply:

$$\frac{\left\langle Tr \left[\sum_{I,J} \psi_{0I}(x) \left(\frac{1}{T} \right)_{IJ} \psi_{0J}^\dagger(x) \right]^2 \right\rangle}{\left\langle \left[Tr \sum_{I,J} \psi_{0I}(x) \left(\frac{1}{T} \right)_{IJ} \psi_{0J}^\dagger(x) \right]^2 \right\rangle} = \frac{5\pi^2 \bar{n}}{\int d\rho d(\rho) \frac{1}{\rho^4}} \simeq 5\pi^2 \bar{n} \bar{\rho}^4 \sim \frac{5}{8}, \quad (29)$$

where we have used the ansatz (20) [21]. Some comments on eq. (29) are in order. First of all, in general quark condensate is rather inhomogeneous, and for parametrically dilute instanton ensemble this ratio is small. However, with empirical diluteness it happens to be not so small, about 0.6. In principle, by measuring the left-hand side and right-hand-side of (29) on the lattice *separately*, one can estimate the accuracy of the universality of the effective mass.

However, since different configurations and even points have different leading instanton, the corresponding value $T_{I^*I^*}$ fluctuates, and the average of its different powers in general leads to different effective masses. (This effect should not be confused with the inhomogeneity of the condensates discussed above.) Let us define a parameter R_m , such that $R_m = 1$ means universal mass $(m_{uu})^2 = m_{uudd}^2$:

$$R_m := \frac{\left\langle Tr \left[\sum_{I,J} \psi_{0I}(x) \left(\frac{1}{T} \right)_{IJ} \psi_{0J}^\dagger(x) \right]^2 \right\rangle}{5\pi^2 \bar{\rho}^4 \bar{n} \left\langle \left[Tr \sum_{I,J} \psi_{0I}(x) \left(\frac{1}{T} \right)_{IJ} \psi_{0J}^\dagger(x) \right]^2 \right\rangle}. \quad (30)$$

III. NUMERICAL STUDY OF THE SINGLE INSTANTON APPROXIMATION.

In general, reliability of the SIA depends on the vacuum diluteness. In this section, we want to establish whether the QCD vacuum with realistic density is actually dilute enough for the leading-instanton to be dominant, at least for some observables.

For this purpose we have performed numerical analysis of several correlation functions, measured in the random instanton liquid model. In such ensemble, the vacuum expectation values are obtained by averaging over configurations of randomly distributed instantons of size $\rho = 1/3 fm$. The contribution from the leading-instanton is evaluated by retaining only the largest term in (15), for each configuration.

We begin by considering two quark condensates χ_{uu} and χ_{uudd} , introduced in (21) and (26). We will show later that they represent all generic observables which receive contribution from one and two zero-mode propagators, respectively.

In this calculation we average 5000 configurations of 20 instantons in a box of volume $3.4 \times 1.8^3 fm^4$. The results of this simulation are presented in table I.

From these results one can see how the accuracy of SIA (keeping only the closest instanton) depends on the particular matrix element being evaluated. Naturally, the accuracy increases with the dimension of the operator involved, because it diminish the contribution of distant instantons. Specifically, SIA for dimension-six local operators which receive contribution from two zero-mode propagators agree with full calculation within a few percent. On the other hand, prediction for operators/correlators with only one zero-mode propagators are not really accurate: the error in quark condensate is large ($\gtrsim 35\%$).

TABLE II: Universality parameter, R_m and the effective masses evaluated in the RILM and in the IILM

Quantity	RILM calculation	IILM calculation
R_m	0.4	0.2
m_{uu}	$120 MeV$	$177 MeV$
$\sqrt{m_{uudd}^2}$	$65 MeV$	$91 MeV$

Next we consider two-point correlation functions. This allows us to determine the scale at which the closest instanton is no longer dominant. At this purpose we have measured the pion pseudo-scalar two point function,

$$P(x) := \langle 0 | J_5(x) J_5^\dagger(0) | 0 \rangle, \quad (31)$$

where,

$$J_5(x) := \bar{u}(x) \gamma_5 d(x). \quad (32)$$

This particular choice is motivated by the fact that such correlation function is known to receive maximal contribution from quark zero-modes [13]. One expects many instantons effects to become important for $|x|$ larger than the instanton size and smaller than the typical distance between two neighbor instantons:

$$1/3fm \lesssim |x| \lesssim 1fm \quad (33)$$

Results of simulations including the contribution from all instantons and from the leading-instanton only are reported in figure (1). One can see that the agreement is lost for rather large values of $|x|$ ($|x| \gtrsim 0.6fm$).

In the last section we saw that the SIA does not only assume leading-instanton dominance, but involves some effective mass parameters, which collectively describe the effects of all other instantons (see eqs. (25) and (28)). In the next section we shall determine numerically such parameters.

IV. NUMERICAL STUDY OF THE QUARK EFFECTIVE MASS PARAMETERS

In section II we argued that the universality of the effective mass, which collectively describes the effects of all non-leading-instantons, can be put in relation to the fluctuations of the quark condensates through eq. (29).

Obviously, the accuracy of calculations in the SIA depends on the value of R_m (defined in (30)) in realistic ensembles.

We have evaluated R_m and the corresponding effective masses, numerically [22] in the random instanton liquid and in the interacting liquid (for a review of these ensembles see [3]). Our results are summarized in table II:

These results show that, in the instanton vacuum with realistic density, the *universality does not hold*

$$m_{uu}^2 \neq m_{uudd}^2. \quad (34)$$

This implies that an effective mass extracted from the quark condensate can not be used in calculations involving more than one zero-mode propagator.

On the other hand, the results of numerical simulations presented in section III have shown that matrix elements involving only one zero-mode propagator (like the quark condensate) can not be reliably evaluated in the SIA, simply because the leading instanton is not dominant. As a consequence, one is forced to consider only correlation functions involving at least two such propagators and therefore m_{uu} is of no practical usefulness.

In more general terms, one may address the question whether the effective mass parameter depends on the particular correlation function being evaluated. If so, this feature would spoil much of the predictive power of the SIA. In such pessimistic scenario the SIA would only allow to work out the functional expressions of small-sized correlations, but not their overall normalization. However we will show that the effective mass parameters depend essentially on the number of zero-mode propagators involved, and that m_{uudd}^2 is in a way universal for a number of applications. In this case, SIA is predictive including the normalization. To check that we have extracted m_2^2 from the analysis of several hadronic two-point functions evaluated in SIA and in the liquid. In particular, we considered the pion pseudo-scalar the scalar diquark and the a nucleon scalar correlation functions:

$$P(x) = \langle 0 | J_5(x) J_5^\dagger(0) | 0 \rangle, \quad (35)$$

$$D(x) = \langle 0 | J_{C5}^a(x) J_{C5}^{a\dagger}(0) | 0 \rangle, \quad (36)$$

$$N(x) = \langle 0 | Tr [\eta(x) \bar{\eta}(0) \gamma_4] | 0 \rangle, \quad (37)$$

TABLE III: Estimates of the quark effective mass m_2^2 from several correlation functions.

Correlation function	$m_2^2 [MeV^2]$ (RILM)	$m_2^2 [MeV^2]$ (IILM)
χ_{uudd} condensate	$(65)^2$	$(91)^2$
pion pseudo-scalar	$(65)^2$	$(105)^2$
diquark scalar	$(69)^2$	$(105)^2$
nucleon scalar	$(67)^2$	$(105)^2$

TABLE IV: Determinantal masses evaluated in the RILM and in the IILM as compared to m_1 and m_2^2 , defined in section IV.

mass	RILM calculation	IILM calculation
m_1	$120 MeV$	$177 MeV$
m_{det}	$63 MeV$	$102 MeV$
m_2^2	$(65 MeV)^2$	$(103 MeV)^2$
m_{det}^2	$(64 MeV)^2$	$(103 MeV)^2$

where,

$$J_5(x) := \bar{u}(x) \gamma_5 d(x). \quad (38)$$

$$J_{C5}^a(x) := \epsilon^{abc} u_b(x) C \gamma_5 d_c(x). \quad (39)$$

$$\eta_\alpha(x) := \epsilon^{abc} (u^a(x) C \gamma_5 u^b(x)) u_\alpha^c(x). \quad (40)$$

All these correlations function are known to receive contribution from two propagators in the zero-mode.

The comparison between results obtained in the SIA, in the random instanton liquid model (RILM) and in the interacting instanton liquid model (IILM) are reported in figs. (2), (3) and (4). The corresponding values for $\sqrt{m_2^2}$ are presented in table (III). These values are indeed rather different from the traditionally adopted estimate $m^* = 170 MeV$, extracted from the quark condensate.

The general reason why these masses are rather small is the following. Instantons have fluctuating strength of interaction with others in the ensemble: some of them are “hermits” and have small matrix elements in the corresponding entries of the overlap matrix T . As in all expressions we average the *inverse* of this matrix, the contribution of such “hermits” is enhanced. This lowers the value of the effective masses. Furthermore, because random ensemble of RILM has more such “hermits”, as compared to IILM (where the fermionic determinant in the statistical weight suppresses them), these masses are smaller in RILM as compared to IILM. Such discrepancy reflects the fact that the two ensembles give actually quite different correlation functions [3].

From these results we conclude that m_2^2 seems to be a universal parameter, describing the collective many-instanton effects.

It is important to know what value of m_2^2 is suggested by the available phenomenology. As before, we chose to consider the pion pseudo-scalar correlator, because it receives maximal contribution from instanton zero-modes. The traditional “pole-plus-continuum” model for the spectral decomposition of $P(x)$, gives [15], [13]:

$$P(x) = \lambda_\pi^2 D(m_\pi; x) + \frac{3}{8\pi^2} \int_{s_0}^{\infty} ds s D(\sqrt{s}; x), \quad (41)$$

where $D(m; x)$ is the scalar propagator, s_0 is the threshold for the continuum ($\sqrt{s_0} \simeq 1.6 GeV$) and the pseudo-scalar decay constant λ_π is given by:

$$\lambda_\pi = \langle 0 | \bar{u} \gamma_5 d | \pi \rangle = \frac{f_\pi m_\pi^2}{m_u + m_d} \simeq (480 MeV)^2. \quad (42)$$

We determined m_2^2 , by fitting the SIA prediction to the phenomenological curve obtained from (41). We found (see fig. 2):

$$m_{2phen.}^2 = (86 MeV)^2. \quad (43)$$

To further check the approach, we have evaluated the scalar proton two-point function $N(x)$, using the value (43) and we have compared with the phenomenological curve (see fig. (4) [23]). In summary: with this value we obtained good very good agreement with phenomenology and therefore we suggest that (43) should be used for the applications of the SIA, when two zero-mode propagators are involved.

V. EVALUATION OF AN EFFECTIVE MASS IN THE FERMIONIC DETERMINANT

The propagator is not the only place where the Dirac operator appears: the QCD statistical sum contains its *determinant*, appearing in power given by the number of light quark flavors N_f . If one considers the academic vacuum with only one instanton, this determinant contains the product of “current” quark masses for all quarks [8]. If this would be the final answer for the instanton density, instanton effect would be strongly suppressed by their small values.

However, in physical vacuum there are sufficiently many instantons to break chiral symmetry and produce non-zero quark condensates and effective quark masses, which substitute for much smaller “current” masses and make instanton effects significantly stronger. The interplay between these effective masses and current quark masses is especially interesting for strange quark, since the former and the latter, m_s , are of comparable magnitude. This issue has been studied e.g. in recent paper [16], where it was concluded that the usual additive formula for total effective quark mass of the strange quark $M_s^{tot} = m_{eff}(m_s = 0) + m_s$ is wrong, and the true value of M_s^{tot} is not very different from that for u,d quarks because $m_{eff}(m_s)$ strongly decreases with m_s .

Apart of the role of strange quark mass in general, there is also a general issue of correct connection of units and vacuum parameters (with the instanton density being one of them) for QCD with different number of flavors. (For example, between no-quark or quenched QCD and the physical world.) In order to study all of this, it is important to know what is the absolute magnitude of the fermionic determinants in the instanton-based vacuum models considered. Some of those are reported in this section.

In the instanton-based model context, the fermionic determinant is usually represented by the determinant of the overlap matrix T (see description e.g. in [3]) in the zero mode subspace. After averaging over appropriate ensemble, one can define the so-called “determinantal masses”:

$$m_{det}^i := \frac{\langle (\det[\overline{D}])^{i/N} \rho^i \rangle}{\langle \rho^i \rangle}, \quad i = 1, 2, \dots \quad (44)$$

where index i refers to number of flavors and N denotes the number of instantons. Their values tell us how much the presence of fermions reduces the instanton density, compared to the same ensemble without them.

Originally, in [4], [14] an estimate for the determinant effective mass was extracted from the averaging of the 't Hooft Lagrangian assuming factorization of quark condensates, and using the same $m^* = 170 \text{ MeV}$. If so, each flavor reduces instanton density by the factor $m^* \bar{\rho} \approx 0.28$. As we will see shortly, the corresponding reduction factor is actually even smaller.

In principle, there is no reason why the values of m_{det} and m_{det}^2 should agree with m_1 and m_2^2 , defined in the previous section: we now average positive rather than negative powers of the overlap matrix.

We have evaluated the determinantal masses in the RILM and in the IILM. Results are reported in table IV. Some comments are in order. First of all note that, in both ensembles, the values of m_{det}^2 turn out to be quite consistent with the values of m_2^2 . Furthermore, the fluctuations of the determinantal mass, m_{det} and $(m_{det})^2$ are very small:

$$m_{det}^2 - (m_{det})^2 \ll m_2^2 - (m_1)^2, \quad (45)$$

implying essentially that m_1 is inconsistent with m_{det} . This fact could have two possible explanations. On the one hand, one could argue that m_1 is a somewhat ill-defined parameter, because the SIA can not be used to evaluate quark condensate. On the other hand, one could observe that larger fluctuations for the effective masses defined in IV should not be surprising, since such parameters appear always in denominators of SIA calculations.

VI. CONCLUSIONS AND OUTLOOK

Summarizing our study of the SIA approximation in QCD, we first notice that this approach has been related to the theory of the full ensemble and all the effective parameters previously loosely called “effective masses” are defined. All of them describe different aspects of collective interaction between the “leading” instanton (the closest to the observation points) and all others, and related to the overlap matrix T . Different effective mass values simply follow from different ensemble averaging. In particular, the factor $\frac{1}{m_1}$, appearing in SIA calculations with one propagator in the zero-mode, does not correspond to the square root of the factor $\frac{1}{m_2^2}$, appearing when two such propagators are involved.

We have made numerical simulations in the RILM and IILM and found that the contribution of the leading-instanton actually dominates all condensates of operators of dimension six or more, as well as short-distance correlation functions ($|x| \lesssim 0.6 fm$). This however is true only for correlation functions with at least two zero-mode propagators involved. Earlier estimates extracted from the quark condensate are not accurate.

Furthermore, the parameter $\frac{1}{m_2^2}$ is approximatively universal for several correlation functions with *two* zero-mode propagators involved. We have also extracted a phenomenological estimate of its value from the analysis of the pion pseudo-scalar correlator. We found $m_{2\text{phen.}}^2 \simeq (86\text{MeV})^2$, much smaller than the value originally obtained from the quark condensate. Our new value should be used in many applications of the SIA.

Finally, we have compared our estimates for the effective mass parameters m_1 and m_2^2 , with the measurements of the “determinantal” masses, introduced in [14]. We observed substantial agreement between m_2^2 and m_{det}^2 both in the RILM and the IILM, but differences imply that light quarks are about twice more effective

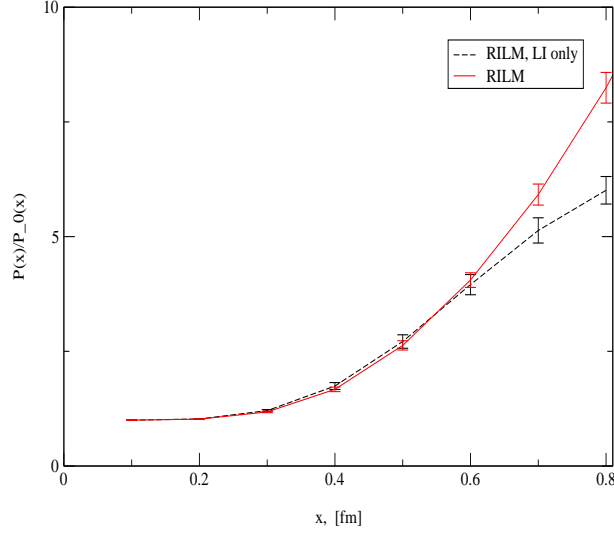


FIG. 1: Pion pseudo-scalar correlation function in the RILM, normalized to the same correlation function in the free theory. The solid line corresponds to the RILM data, and the dashed line represents the RILM, LI only contribution.

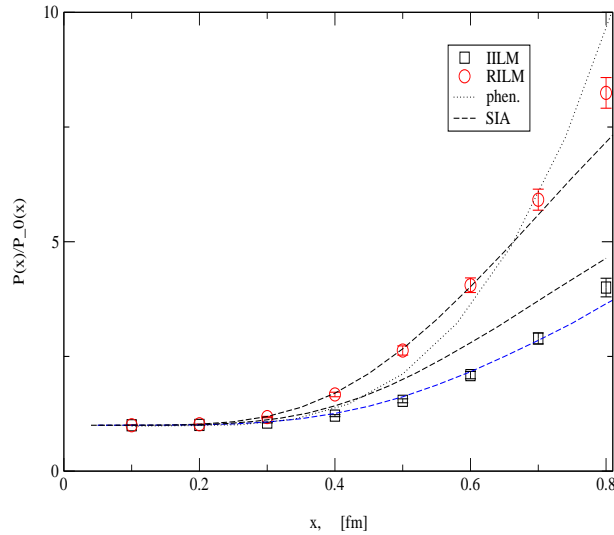


FIG. 2: Pion pseudo-scalar two-point function normalized to the same correlation function in the free theory. The open circles (squares) represent RILM (IILM) points, the dashed lines represent SIA calculations with masses given in table III and the dotted line is the phenomenological curve obtained from the spectral decomposition.

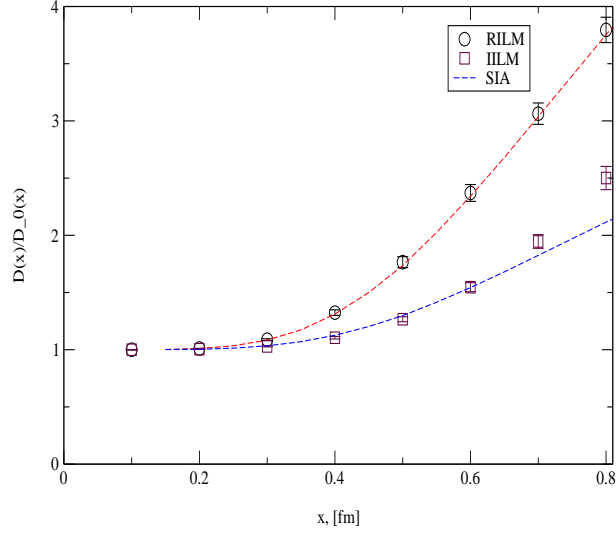


FIG. 3: Diquark scalar two-point function normalized to the same correlation function in the free theory. The open circles (squares) represent RILM (IILM) points and the dashed lines represent SIA calculations with the effective masses given in table III

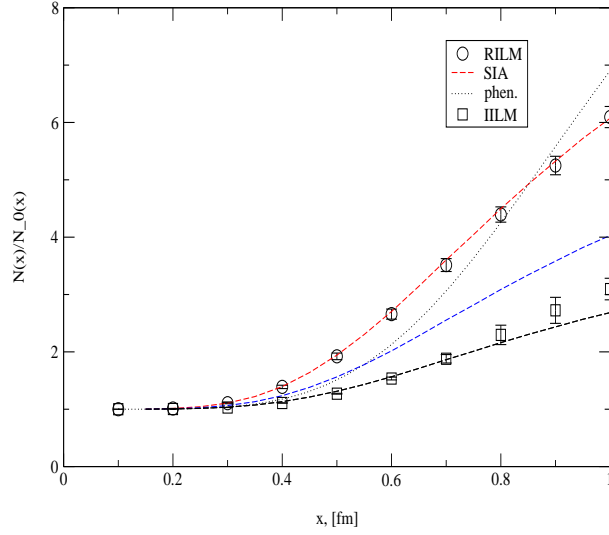


FIG. 4: Nucleon scalar two-point function normalized to the same correlation function in the free theory. The open circles (squares) represent RILM (IILM) points and the dashed lines represent SIA calculations with the effective masses given in table III.

Acknowledgments

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- [16] M. Musakhanov, "Current mass dependence of the quark condensate and the constituent quark mass," hep-ph/0104163.
- [17] For simplicity, we shall often use the term "instanton" to denote instantons and/or anti-instantons.
- [18] For sake of simplicity, we are explicitly dropping all collective coordinates, except the instanton position z_i ; moreover the use of the singular gauge is assumed everywhere.
- [19] Notice that for all gauge invariant matrix elements, the average over the color orientation is trivial.
- [20] Here, we have neglected all small current quark mass terms in $1/T$.
- [21] Alternatively, we repeated the calculation using a parameterization of the lattice measurements of $d(\rho)$, which is peaked about somewhat higher values of ρ ($\rho \simeq 3.9$). Both calculation give basically the same result.
- [22] Here we have averaged on 5000 configuration of 256 instantons in a $4^4 fm^4$ box.
- [23] Notice that, in this case, the pole contribution depends on the nucleon "decay-constant" Λ_s , which is not known experimentally. We have therefore used an estimate ($\Lambda_s = 2.5 fm^{-3}$) which reasonably agrees with several model calculations [3]